# Deriving the Explicit Guidance Equations

These are my notes to understand the powered explicit guidance algorithm from NASA. The original can be found at https://ntrs.nasa.gov/citations/19660006073.

## Coordinates

The vector $\vec{r}$ points from the centre of the planet to the spacecraft. We work in a rotating radial, normal, circumferential coordinate system

$$\hat{e}_r := \frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{r}, \qquad \hat{e}_z := \frac{\vec{r} \times \dot{\vec{r}}}{\left\|\vec{r} \times \dot{\vec{r}}\right\|}, \qquad \hat{e}_\varphi := \hat{e}_z \times \hat{e}_r. \tag{1}$$

This essentially means we are using cylindrical coordinates where the cylinder is constantly rotating such that the $\vec{r}$-$\dot{\vec{r}}$-plane is the $z = 0$ plane. Because of this the velocity $\dot{\vec{r}}$ never has any $\hat{e}_z$ component, but there can be acceleration in this direction, causing the entire coordinate system to rotate. Furthermore this implies that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r$ also only has components in $\hat{e}_r$ and $\hat{e}_\varphi$, and to be more precise only in the latter direction, because otherwise its magnitude would change. The explicit terms for the time-derivatives of the base vectors are

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r = \frac{\hat{e}_\varphi \cdot \dot{\vec{r}}}{r}\hat{e}_\varphi, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z = -\frac{\hat{e}_z \cdot \ddot{\vec{r}}}{\hat{e}_\varphi \cdot \dot{\vec{r}}}\hat{e}_\varphi, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi = -\frac{\hat{e}_\varphi \cdot \dot{\vec{r}}}{r}\hat{e}_r + \frac{\hat{e}_z \cdot \ddot{\vec{r}}}{\hat{e}_\varphi \cdot \dot{\vec{r}}}\hat{e}_z. \tag{2}$$

The derivations are in Appendix A.

## Velocity and Acceleration

The velocity $\dot{\vec{r}}$ can be written as

$$\dot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} \overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r \overset{(2)}{=} \dot{r}\hat{e}_r + \left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi. \tag{3}$$

Using this we can write the $\hat{e}_r$-component of the acceleration $\ddot{\vec{r}}$ as

$$\begin{aligned} \hat{e}_r \cdot \ddot{\vec{r}} &= \hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\dot{\vec{r}} \overset{(3)}{=} \hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{r}\hat{e}_r + \left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi\right] = \ddot{r} + \left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_r \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \\ &= \ddot{r} - \frac{\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)^2}{r}\hat{e}_r. \end{aligned} \tag{4}$$

## Time

The current time is always 0 and the time at the target state is $t$. So the current position and velocity is $\vec{r}(0), \dot{\vec{r}}(0)$ and the target position and velocity is $\vec{r}(t), \dot{\vec{r}}(t)$.

## Rocket Engine Acceleration

The acceleration due to the rocket engine is $a\hat{f}$ where $a$ is the magnitude of the acceleration and $\hat{f}$ is the unit vector pointing from the rocket engine to the centre of mass of the spacecraft. Note that the rocket engine is operating at constant thrust (constant force) $F_{\text{engine}}$ while the mass $m$ of the rocket reduces (due to fuel consumption). With $R$ as the constant rate of fuel consumption we can write $m(t) = m(0) - Rt$. The engine acceleration is then

$$a(t) = \frac{F_{\text{engine}}}{m(t)} = \frac{F_{\text{engine}}}{m(0) - Rt} = \frac{a(0)}{1 - \frac{t}{\tau}} \tag{5}$$

with $a(0) = \frac{F_{\text{engine}}}{m(0)}$ and $\tau = \frac{m(0)}{R}$. Note that also $v_e = a(0)\tau$ where $v_e$ is the average exhaust velocity relative to the spacecraft (often called $I_{\text{sp}}$). The derivation for this is shown in Appendix B.

## Net Acceleration

There are just two forces acting on the spacecraft - gravity and the thrust of the rocket engine. The net acceleration is the sum of both

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{e}_r + a\hat{f}, \tag{6}$$

where $a$ is the acceleration due to the thrust of the rocket engine and $\hat{f}$ is the unit vector pointing from the rocket engine to the centre of mass of the spacecraft. Inserting this into Equation (4) gives us

$$\ddot{r} + \frac{\mu}{r^2} - \frac{\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)^2}{r} = a\hat{f} \cdot \hat{e}_r. \tag{7}$$

## Pitch Steering

For the pitch steering we choose the Ansatz

$$\hat{f} \cdot \hat{e}_r = A + Bt + \frac{1}{a}\frac{\mu}{r^2} - \frac{1}{a}\frac{\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)^2}{r} \tag{8}$$

with constants $A$ and $B$. We insert this into Equation (7) and get

$$\ddot{r}(t) = (A + Bt)a(t). \tag{9}$$

Integrating this equation twice gives us

$$\begin{aligned} \dot{r}(t) &= \dot{r}(0) + b_0(t)A + b_1(t)B \\ r(t) &= r(0) + \dot{r}(0)t + c_0(t)A + c_1(t)B \end{aligned} \tag{10}$$

with

$$\begin{aligned} b_0(t) &= \int_0^t a(s)\,\mathrm{d}s = -v_e \ln\left(1 - \frac{t}{\tau}\right) \\ b_n(t) &= \int_0^t s^n a(s)\,\mathrm{d}s = b_{n-1}(t)\tau - \frac{v_e t^n}{n} \\ c_0(t) &= \int_0^t b_0(s)\,\mathrm{d}s = b_0(t)t - b_1(t) \\ c_n(t) &= \int_0^t b_n(s)\,\mathrm{d}s = c_{n-1}(t)\tau - \frac{v_e t^{n+1}}{n(n+1)}. \end{aligned} \tag{11}$$

The derivations for Equation (11) can be found in Appendix C.

## A Derivatives of the Base Vectors

We calculate $\hat{e}_\varphi$ more explicitly via

$$\hat{e}_\varphi := \hat{e}_z \times \hat{e}_r \overset{(1)}{=} \frac{\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r}}{r\left\|\vec{r} \times \dot{\vec{r}}\right\|} = \frac{r^2\dot{\vec{r}} - \left(\vec{r} \cdot \dot{\vec{r}}\right)\vec{r}}{r\left\|\vec{r} \times \dot{\vec{r}}\right\|} = \frac{\dot{\vec{r}} - \left(\hat{e}_r \cdot \dot{\vec{r}}\right)\hat{e}_r}{\left\|\hat{e}_r \times \dot{\vec{r}}\right\|} = \frac{\dot{\vec{r}} - \left(\hat{e}_r \cdot \dot{\vec{r}}\right)\hat{e}_r}{\hat{e}_\varphi \cdot \dot{\vec{r}}} \tag{12}$$

where we have used the triple product expansion and in the last step we used the fact that $\|\hat{e}_1 \times \vec{v}\| = \hat{e}_2 \cdot \vec{v}$ for any vector $\vec{v}$ that has only components in $\hat{e}_1$ and $\hat{e}_2$ and $\hat{e}_1 \perp \hat{e}_2$. Note that rearranging Equation (12) confirms that the velocity has no $\hat{e}_z$-component

$$\dot{\vec{r}} \overset{(12)}{=} \left(\hat{e}_r \cdot \dot{\vec{r}}\right)\hat{e}_r + \left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi. \tag{13}$$

Another way to calculate the velocity is

$$\dot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r. \tag{14}$$

Looking at Equations (13) and (14) we can also confirm that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r$ only points towards $\hat{e}_\varphi$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r = \frac{1}{r}\underbrace{\left(\hat{e}_r \cdot \dot{\vec{r}} - \dot{r}\right)}_{=0}\hat{e}_r + \frac{1}{r}\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi = \frac{1}{r}\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi. \tag{15}$$

The (unit-mass) angular momentum $\vec{r} \times \dot{\vec{r}}$ changes as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{r} \times \dot{\vec{r}}\right) = \underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \ddot{\vec{r}} \overset{(1)}{=} r\vec{e}_r \times \ddot{\vec{r}} \tag{16}$$

and its magnitude changes as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{r} \times \dot{\vec{r}}\right\| \overset{(1),\,(12)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(r\hat{e}_\varphi \cdot \dot{\vec{r}}\right) = \dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} + r\hat{e}_\varphi \cdot \ddot{\vec{r}}. \tag{17}$$

Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r} \times \dot{\vec{r}}}{\left\|\vec{r} \times \dot{\vec{r}}\right\|} &\overset{(16),\,(17)}{=} \frac{r\hat{e}_r \times \ddot{\vec{r}}}{r\hat{e}_\varphi \cdot \dot{\vec{r}}} - \frac{\left(\dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} + r\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)r\hat{e}_r \times \dot{\vec{r}}}{\left(r\hat{e}_\varphi \cdot \dot{\vec{r}}\right)^2} \\
&\overset{(13)}{=} \frac{r\hat{e}_r \times \ddot{\vec{r}}}{r\hat{e}_\varphi \cdot \dot{\vec{r}}} - \frac{\left(\dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} + r\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\overbrace{\hat{e}_r \times \hat{e}_\varphi}^{=\hat{e}_z}}{r\left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)^2} \\
&= \frac{r\hat{e}_r \times \left(\left(\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)\hat{e}_\varphi + \left(\hat{e}_z \cdot \ddot{\vec{r}}\right)\hat{e}_z\right)}{r\hat{e}_\varphi \cdot \dot{\vec{r}}} - \frac{\left(\dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} + r\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)\hat{e}_z}{r\hat{e}_\varphi \cdot \dot{\vec{r}}} \\
&= \frac{r\left(\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)\hat{e}_z - r\left(\hat{e}_z \cdot \ddot{\vec{r}}\right)\hat{e}_\varphi - \left(\dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} + r\hat{e}_\varphi \cdot \ddot{\vec{r}}\right)\hat{e}_z}{r\hat{e}_\varphi \cdot \dot{\vec{r}}} \\
&= \frac{-r\left(\hat{e}_z \cdot \ddot{\vec{r}}\right)\hat{e}_\varphi - \left(\dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}}\right)\hat{e}_z}{r\hat{e}_\varphi \cdot \dot{\vec{r}}}.
\end{aligned} \tag{18}$$

Because $\hat{e}_z$ does not change its magnitude, we know that $\hat{e}_z \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z = 0$ and therefore we can infer from Equation (18) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z = -\frac{\hat{e}_z \cdot \ddot{\vec{r}}}{\hat{e}_\varphi \cdot \dot{\vec{r}}}\hat{e}_\varphi, \tag{19}$$

showing that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z$ has no $\hat{e}_r$-component as well and points solely into the $\hat{e}_\varphi$-direction. Due to the same reason we know that

$$\begin{aligned}
0 &\overset{(18)}{=} \dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right) \cdot \dot{\vec{r}} \\
&= \dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left[\left(\hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right)\hat{e}_r + \left(\hat{e}_z \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right)\hat{e}_z\right] \cdot \left[\left(\hat{e}_r \cdot \dot{\vec{r}}\right)\hat{e}_r + \left(\hat{e}_\varphi \cdot \dot{\vec{r}}\right)\hat{e}_\varphi\right] \\
&= \dot{r}\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right)\underbrace{\left(\hat{e}_r \cdot \dot{\vec{r}}\right)}_{=\dot{r}} = \dot{r}\left[\hat{e}_\varphi \cdot \dot{\vec{r}} + r\left(\hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right)\right] \\
\implies \hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi &= -\frac{\hat{e}_\varphi \cdot \dot{\vec{r}}}{r}.
\end{aligned} \tag{20}$$

The $\hat{e}_z$-component of $\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi$ can be calculated via

$$\begin{aligned}
\hat{e}_z \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi &\overset{(1)}{=} \hat{e}_z \cdot \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z\right) \times \hat{e}_r + \hat{e}_z \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r\right)\right] \\
&= \hat{e}_z \cdot \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z\right) \times \hat{e}_r\right) + \underbrace{\hat{e}_z \cdot \left(\hat{e}_z \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r\right)\right)}_{=0} \overset{(19)}{=} \frac{\hat{e}_z \cdot \ddot{\vec{r}}}{\hat{e}_\varphi \cdot \dot{\vec{r}}},
\end{aligned} \tag{21}$$

so overall

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi &= \left(\hat{e}_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\varphi\right)\hat{e}_r + \left(\hat{e}_z \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z\right)\hat{e}_z \\
&\overset{(20),\,(21)}{=} -\frac{\hat{e}_\varphi \cdot \dot{\vec{r}}}{r}\hat{e}_r + \frac{\hat{e}_z \cdot \ddot{\vec{r}}}{\hat{e}_\varphi \cdot \dot{\vec{r}}}\hat{e}_z
\end{aligned} \tag{22}$$

which concludes our derivations of the time derivates of all base vectors □.

## B Engine Exhaust Velocity

First our spacecraft has mass $m$ and velocity $v$. One infinitesimal moment later it will have accelerated fuel of mass $\mathrm{d}m$ out the back with a relative velocity of $-v_e$, or an absolute velocity $v - v_e$. Due to Newton's third law the rocket with a new mass of $m - \mathrm{d}m$ feels a forward acceleration where it gains the infinitesimal velocity $\mathrm{d}v$, making its new velocity $v + \mathrm{d}v$. Do to momentum conservation we have

$$\begin{aligned}
mv &= (m - \mathrm{d}m)(v + \mathrm{d}v) + \mathrm{d}m(v - v_e) \\
&= mv - v\,\mathrm{d}m + m\,\mathrm{d}v - \underbrace{\mathrm{d}m\,\mathrm{d}v}_{\approx 0} + v\,\mathrm{d}m - v_e\,\mathrm{d}m
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\implies v_e\,\mathrm{d}m = m\,\mathrm{d}v \\
&\implies v_e\underbrace{\frac{\mathrm{d}m}{\mathrm{d}t}}_{=R} = m\underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}}_{=a} \\
&\implies v_e = a\frac{m}{R} = a\tau \quad \square.
\end{aligned} \tag{24}$$

## C Derivation of the Thrust Integrals and their Relations

We are interested in solving the integrals of the form

$$b_n(t) = \int_0^t s^n a(s)\,\mathrm{d}s, \quad c_n(t) = \int_0^t b_n(s)\,\mathrm{d}s \tag{25}$$

where $a$ is the rocket engine acceleration (5). Through substitution we reach

$$b_n(t) = \int_0^t s^n a(s)\,\mathrm{d}s \overset{(5),\,(24)}{=} \frac{v_e}{\tau}\int_0^t \frac{s^n}{1-\frac{s}{\tau}}\,\mathrm{d}s = v_e\tau^n \int_0^{\frac{t}{\tau}} \frac{x^n}{1-x}\,\mathrm{d}x = v_e\tau^n I_n\left(\frac{t}{\tau}\right) \tag{26}$$

with

$$I_n(r) := \int_0^r \frac{x^n}{1-x}\,\mathrm{d}x = \int_{1-r}^1 \frac{(1-x)^n}{x}\,\mathrm{d}x. \tag{27}$$

Order 0 can be directly solved

$$I_0(r) = \int_{1-r}^1 \frac{1}{x}\,\mathrm{d}x = \ln(1) - \ln(1-r) = -\ln(1-r) \tag{28}$$

which inserted into Equation (26) gives us the desired result $b_0(t) = -v_e \ln\left(1-\frac{t}{\tau}\right)$. For orders $n > 0$ we find the recursive relationship

$$I_n(r) - I_{n-1}(r) = -\frac{r^n}{n}. \tag{29}$$

Proof:

$$\begin{aligned} I_n(r) &= \int_{1-r}^1 \frac{(1-x)^n}{x}\,\mathrm{d}x \overset{(32)}{=} \sum_{k=0}^{n}\binom{n}{k}\int_{1-r}^1 \frac{(-x)^k}{x}\,\mathrm{d}x \\ &= -\ln(1-r) + \sum_{k=1}^{n}\binom{n}{k}(-1)^k \int_{1-r}^1 x^{k-1}\,\mathrm{d}x \\ &= -\ln(1-r) + \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}\left(1-(1-r)^k\right) \end{aligned} \tag{30}$$

$$\begin{aligned} I_n(r) - I_{n-1}(r) &= \sum_{k=1}^{n-1}\left(\binom{n}{k} - \binom{n-1}{k}\right)\frac{(-1)^k}{k}\left(1-(1-r)^k\right) + \frac{(-1)^n}{n}\left(1-(1-r)^n\right) \\ &\overset{(33)}{=} \sum_{k=0}^{n-1}\binom{n}{k}\frac{(-1)^k}{n}\left(1-(1-r)^k\right) + \frac{(-1)^n}{n}\left(1-(1-r)^n\right) \\ &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(1-(1-r)^k\right) \\ &= -\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(r-1)^k 1^{n-k} \\ &\overset{(32)}{=} -\frac{r^n}{n}, \end{aligned} \tag{31}$$

where we have used

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \tag{32}$$

and

$$\frac{1}{k}\left(\binom{n}{k} - \binom{n-1}{k}\right) = \frac{1}{k}\left(\frac{n!}{k!(n-k)!} - \frac{(n-1)!}{k!(n-1-k)!}\right)$$
$$= \frac{1}{k}\left(\frac{n!}{k!(n-k)!} - \frac{n!}{k!(n-k)!}\frac{n-k}{n}\right)$$
$$= \frac{1}{k}\binom{n}{k}\left(1 - \frac{n-k}{n}\right)$$
$$= \frac{1}{n}\binom{n}{k}. \tag{33}$$

We use this recursive relationship and apply it to our first type of thrust integral (26)

$$\begin{aligned} b_n(t) = v_e\tau^n I_n\left(\frac{t}{\tau}\right) \overset{(29)}{=} v_e\tau^n\left(I_{n-1}\left(\frac{t}{\tau}\right) - \frac{1}{n}\frac{t^n}{\tau^n}\right) = v_e\tau^{n-1}I_{n-1}\left(\frac{t}{\tau}\right)\tau - \frac{v_e t^n}{n} \\ = b_{n-1}(t)\tau - \frac{v_e t^n}{n}. \end{aligned} \tag{34}$$

Now we take a look at

$$c_n(t) \overset{(25)}{=} \int_0^t b_n(s)\,\mathrm{d}s \overset{(26)}{=} v_e\tau^n\int_0^t I_n\left(\frac{s}{\tau}\right)\mathrm{d}s = v_e\tau^{n+1}\int_0^{\frac{t}{\tau}} I_n(x)\,\mathrm{d}x. \tag{35}$$

Using partial integration we rewrite the integral as

$$\begin{aligned} \int_0^r I_n(x)\,\mathrm{d}x = I_n(r)r - \int_0^r xI_n{}'(x)\,\mathrm{d}x \overset{(27)}{=} I_n(r)r - \int_0^r x\frac{x^n}{1-x}\,\mathrm{d}x \\ = I_n(r)r - I_{n+1}(r). \end{aligned} \tag{36}$$

We insert this into the second type of thrust integral (35) and get

$$c_n(t) = v_e\tau^{n+1}I_n\left(\frac{t}{\tau}\right)\frac{t}{\tau} - v_e\tau^{n+1}I_{n+1}\left(\frac{t}{\tau}\right) \overset{(26)}{=} b_n(t)t - b_{n+1}(t). \tag{37}$$

Applying this to the $n = 0$ case gives us $c_0(t) = b_0(t)t - b_1(t)$. Lastly for the $n > 0$ case we insert the first recursive relationship (34) into the second type of thrust integral which yields

$$c_n(t) \overset{(25)}{=} \int_0^t b_n(s)\,\mathrm{d}s \overset{(34)}{=} \tau\int_0^t b_{n-1}(s)\,\mathrm{d}s - \frac{v_e}{n}\int_0^t s^n\,\mathrm{d}s = \tau c_{n-1}(t) - \frac{v_e}{n(n+1)}t^{n+1} \tag{38}$$

which finally concludes our derivations for this chapter □.

# D Inserting the Results

When we know our target state $r(t), \dot{r}(t)$, the time $t$ at which we should reach it, and the current state $r(0), \dot{r}(0)$ then we can calculate the thrust integrals $b_0(t), b_1(t), c_0(t), c_1(t)$ and with those we can calculate $A$ and $B$ via the equations

$$\begin{aligned} \dot{r}(t) = Ab_0(t) + Bb_1(t) + \dot{r}(0) \\ r(t) = Ac_0(t) + Bc_1(t) + \dot{r}(0)t + r(0). \end{aligned} \tag{39}$$

$$
\begin{aligned}
B &= \frac{1}{b_1(t)}(\dot{r}(t) - Ab_0(t) - \dot{r}(0)) \\
\implies Ab_1(t)c_0(t) + (\dot{r}(t) - Ab_0(t) - \dot{r}(0))c_1(t) &= b_1(t)(r(t) - \dot{r}(0)t - r(0)) \\
\iff A(b_1(t)c_0(t) - b_0(t)c_1(t)) &= b_1(t)(r(t) - \dot{r}(0)t - r(0)) + (\dot{r}(0) - \dot{r}(t))c_1(t) \\
\iff A &= \frac{b_1(t)(r(t) - \dot{r}(0)t - r(0)) + (\dot{r}(0) - \dot{r}(t))c_1(t)}{b_1(t)c_0(t) - b_0(t)c_1(t)}.
\end{aligned}
\tag{40}
$$

## E The Equation in Question

The unit vector $\hat{R}_m$ points towards our target and it is constant, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\hat{R}_m = 0$. Therefore